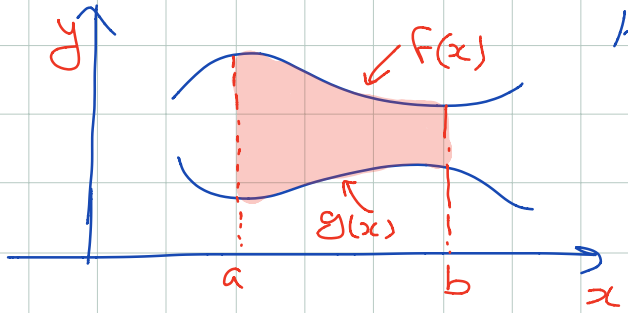
6. Applications of the integral (Volume, averages, work, total mass, ...)

6.1. Area between two curves

Suppose you want to find the shaded area.



How should we do it?

Answer: Area between the graphs

= Area under graph of $F(x)$ − Area under graph of $g(x)$.

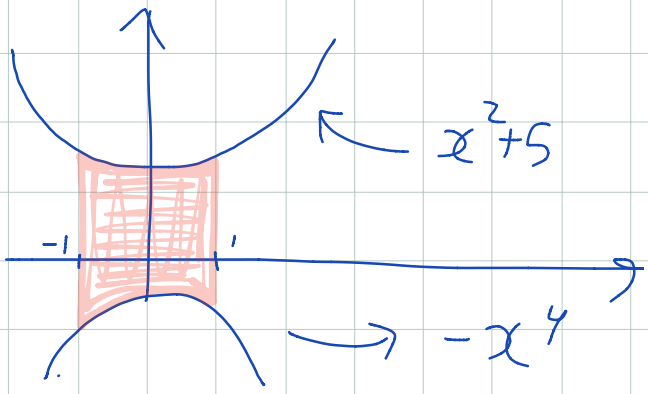
$$= \int_a^b F(x)\,dx - \int_a^b g(x)\,dx = \int_a^b F(x) - g(x)\,dx$$

This is true in the case when $F(x) \geq g(x)$

Example: Find the area between the graphs of

$$F(x) = x^2 + 5 \quad \& \quad g(x) = -x^4, \quad -1 \leq x \leq 1$$

Soln: ① Sketch



② Setup the integral:

$$\text{Area} = \int_{-1}^{1} f(x) - g(x)\,dx = \int_{-1}^{1} x^2 + 5 - (-x^4)\,dx$$

$$= \int_{-1}^{1} x^4 + x^2 + 5\,dx = \frac{x^5}{5} + \frac{x^3}{3} + 5x \Big|_{-1}^{1}$$

$$= \frac{1}{5} + \frac{1}{3} - 5 + \left(-\frac{1}{5} - \frac{1}{3} + 5\right) = \frac{16}{15} + 10 = \frac{166}{15}$$

Example: Find the area between the graph of

$$f(x) = x^2 - 5x - 7,\ g(x) = x - 12,\ x \in [-2, 5]$$

① Sketch (and find points of intersection)

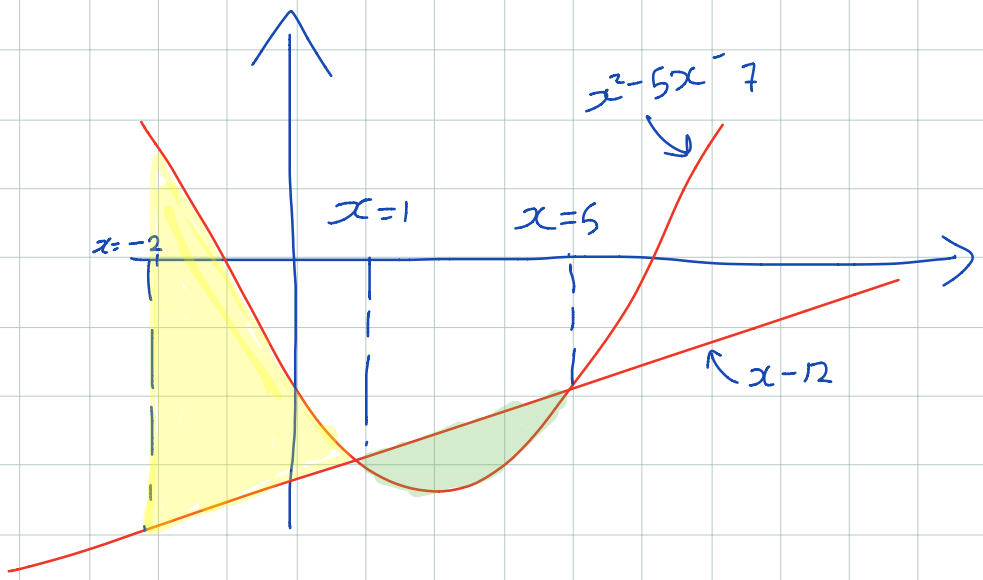
If there are points of intersection, we find them by solving $f(x) = g(x)$, or $f(x) - g(x) = 0$

In this case: $x^2 - 5x - 7 - x + 12 = 0$

$\Leftrightarrow x^2 - 6x - 5 = 0$

$\Leftrightarrow (x-5)(x-1) = 0$

So, when $x = 5$ & $x = 1$, the graphs intersect.



$$\text{Area} = \int_{-2}^{5} y_{\text{Top}} - y_{\text{bottom}}\, dx$$

$$= \int_{-2}^{1} f(x) - g(x)\, dx + \int_{1}^{5} g(x) - f(x)\, dx$$

$$= \int_{-2}^{1} (x^2 - 5x - 7 - x + 12)\, dx + \int_{1}^{5} (x - 12 - x^2 + 5x + 7)\, dx$$

$$= \int_{-2}^{1} x^2 - 6x + 5\, dx + \int_{1}^{5} -x^2 + 6x - 5\, dx$$

$$= \frac{1}{3}x^3 - 3x^2 + 5x \Big|_{-2}^{1} + \left(-\frac{x^3}{3} + 3x^2 - 5x\right)\Big|_{1}^{5}$$

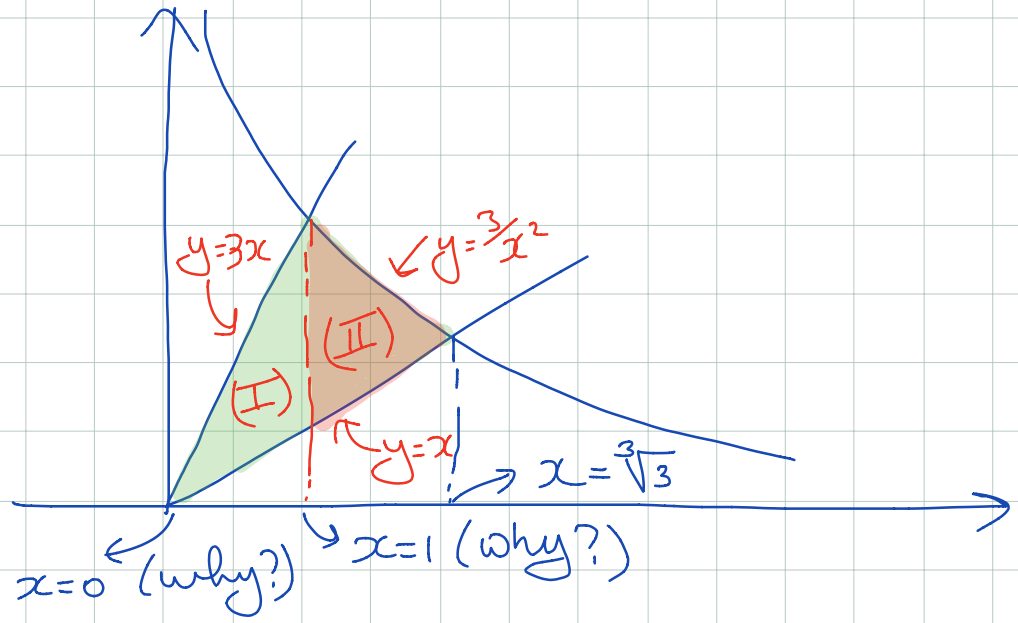
$$= \ldots = 113/3$$

Example 3 / Exercise

Area of the region bounded by the graphs of

$y = 3/x^2$, $y = 3x$ & $y = x$

① Sketch (& find pts of intersection)



Soln: Required Area = Area (I) + Area (II)

$$= \int_0^1 3x - x \, dx + \int_1^{\sqrt[3]{3}} \frac{3}{x^2} - x \, dx$$

$$= \ldots \text{ (finish this)}$$

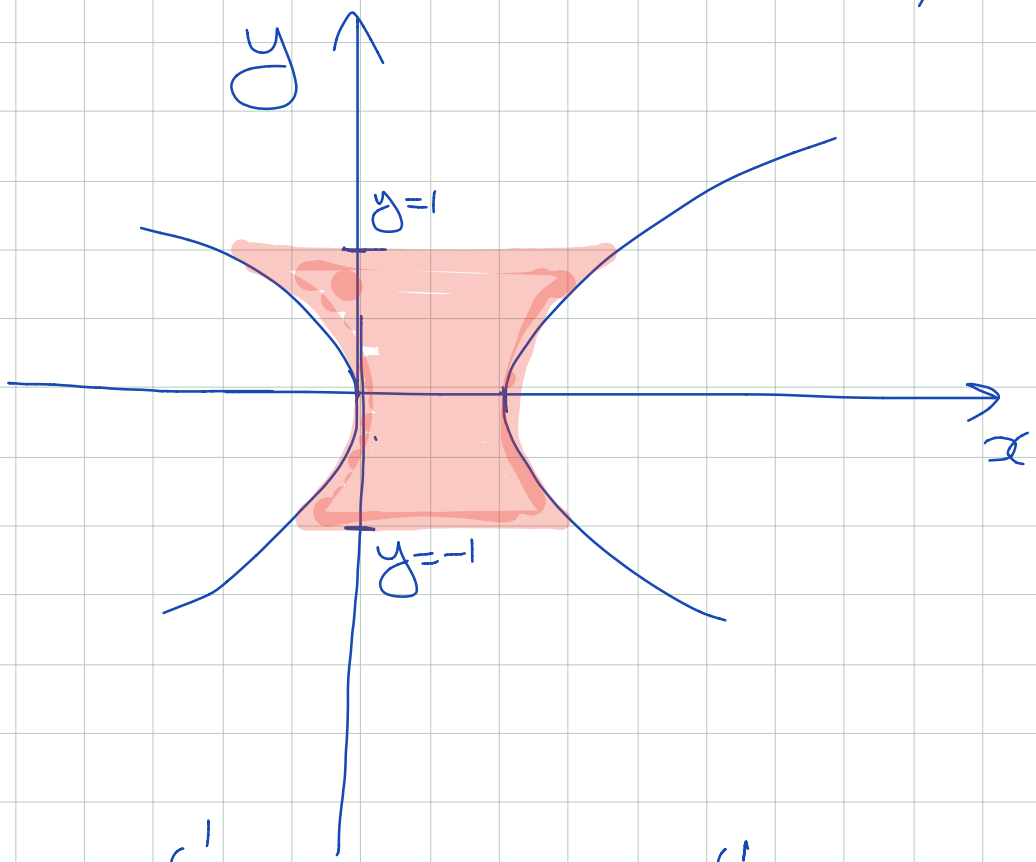
Integration along y-axis:

(Same thing really)

Example: Find the area between the graphs of

$$f(y) = y^2+1 \quad \& \quad g(y) = -y^2, \quad -1 \le y \le 1$$

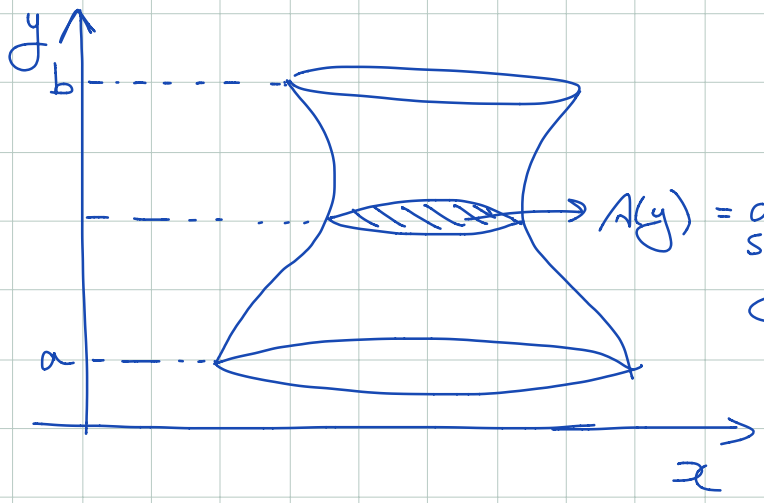
Soln: Sketch (+ find pts of intersection)



$$\text{Area} = \int_{-1}^{1} x_{\text{right}} - x_{\text{left}}\, dx = \int_{-1}^{1} y^2+1-(-y^2)\, dy$$

$$= \int_{-1}^{1} 2y^2+1\, dy = \frac{2y^3}{3} + y \Big|_{-1}^{1} = 2/3+1-(-2/3-1) = 10/3$$

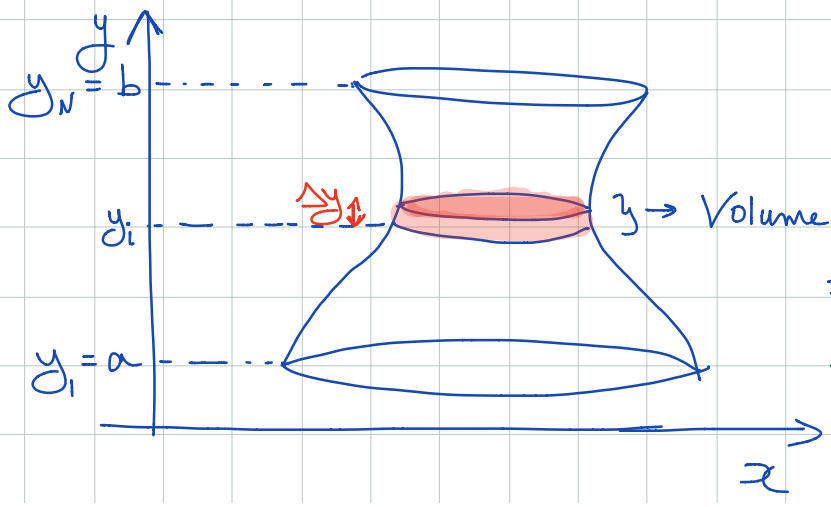
6.2) Setting up integrals: Volume / Density / Ave. Value



Claim: I claim that the Volume is

$$V = \int_a^b A(y)\,dy$$

Why does this make sense?



y → Volume of this slice is

$\approx A(y_i)\,\Delta y$

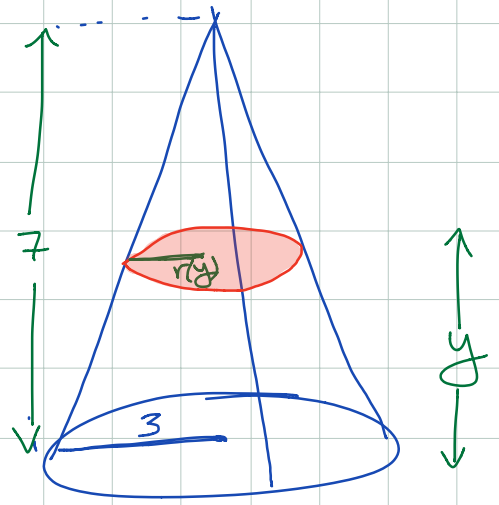
Area of base x height

Total Volume $\approx \sum_{i=1}^{N} A(y_i)\,\Delta y$ ($\approx$ sum of volume of N slices)

$\xrightarrow{\text{make the slices thinner (take limit as } N \to \infty)}$ $V = \int_{y=a}^{b} A(y)\,dy$

Example: Find the volume of a right circular cone with base radius = 3 and height = 7

Soln: ① Sketch



cross-sectional area

② Find the area as a function of y

By similar triangles: $\frac{3}{r} = \frac{7}{7-y}$ (why?)

$\Rightarrow 7r = 21 - 3y \Rightarrow r = 3 - \frac{3}{7}y$

$\Rightarrow A(y) = \pi r^2 = \pi\left(3 - \frac{3}{7}y\right)^2$

③ Integrate: The Volume is $V = \int_0^7 A(y)\,dy$

$$= \int_0^7 \pi(3 - \tfrac{3}{7}y)^2\,dy$$

$$= \cdots = 21\pi$$

Sanity Check: $V(\text{Right Circ. Cone}) = \frac{\pi r^2 h}{3}$

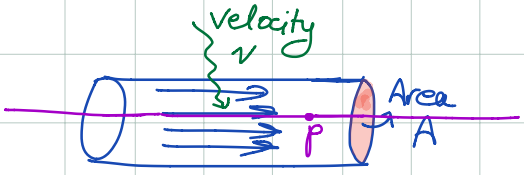
$$= \pi\frac{9 \cdot 7}{3} = 21\pi.$$

Exercises: Do examples 1, 2, 3 in Section 6.2

—×—

Other examples

Flow Rate := Volume/Time of fluid through a tube



I claim: ① If the velocity is constant

Flow rate: $Q = A \times v$

$= \pi r^2 v$ (e.g. m^3/sec)

Justification: a particle flows past P within a 1 second interval if it is at most "1 sec away"

1 sec away $\longleftrightarrow$ velocity (in m/sec)

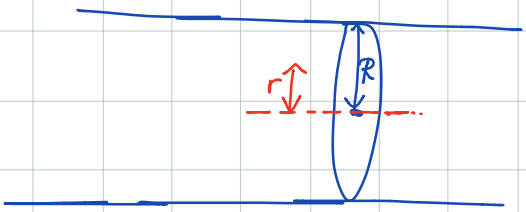
$\Rightarrow$

Volume of fluid flow/sec = Area of fluid flow × velocity

② Otherwise if the flow is laminar

Flow rate $\approx 2\pi \int_0^R r v(r)\, dr$

Velocity varies with r (radially)



Justification: Total flow rate =

$$\sum_{i=1}^{N} \text{flow rates along concentric rings of width } \Delta r$$

$$= \sum A(\text{ring})\, v(\text{ring})$$

$$\approx 2\pi r_i \Delta r \; v(r_i)$$

$$\xrightarrow{\text{limits}} 2\pi \int_0^R r v(r)\, dr$$

Average/mean value

Ave. value of $f(x)$ over $[a, b]$ is $\frac{1}{b-a} \int_a^b f(x)\, dx$

Example Find the average value of $x + \sin x$ over the interval $[0, 2\pi]$

Soln:

$$\text{Ave value} = \frac{1}{2\pi - 0}\int_0^{2\pi} x + \sin x\, dx$$

$$= \frac{1}{2\pi}\cdot\left[\frac{x^2}{2} - \cos x\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{4\pi^2}{2} - 1 - (0 - 1)\right]$$

$$= \frac{2\pi^2}{2\pi} = \pi$$

Mean Value Theorem for integrals

In words: A continuous fcn always takes its average value somewhere in the interval

In math:

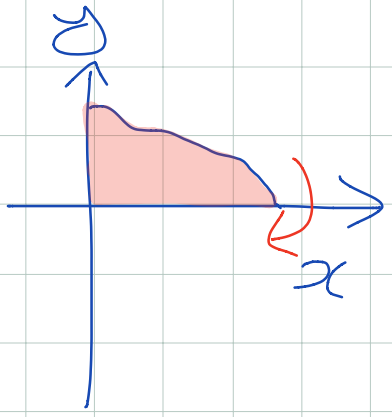
If $f(x)$ is cont. on $[a,b]$, then there is a pt. $c \in [a,b]$ such that

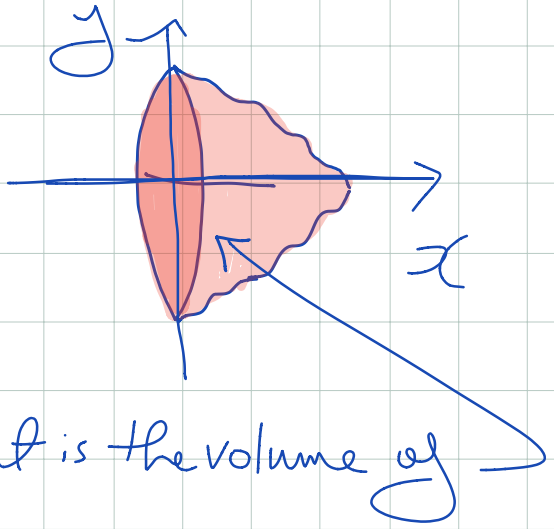
$$f(c) = \frac{1}{b-a}\int_a^b f(x)\, dx$$

Exercise: read the other examples in the book.

6.3) Volume of Revolution

Idea:





what is the volume of

From the previous section:

$V = \int_a^b A(x)\,dx$ but the cross sections are circles of radius $f(x)$

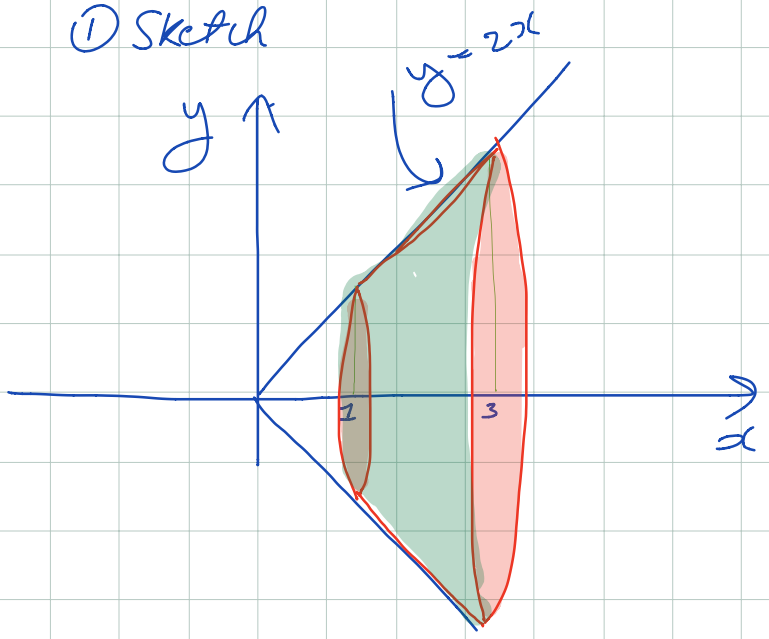
So

$$V = \int_a^b \pi f(x)^2\,dx$$

Volume of Revolution (Disk Method)

Example: Calculate the volume obtained by rotating the region $y = 2x$ about the x-axis for $1 \leq x \leq 3$

Soln ① Sketch

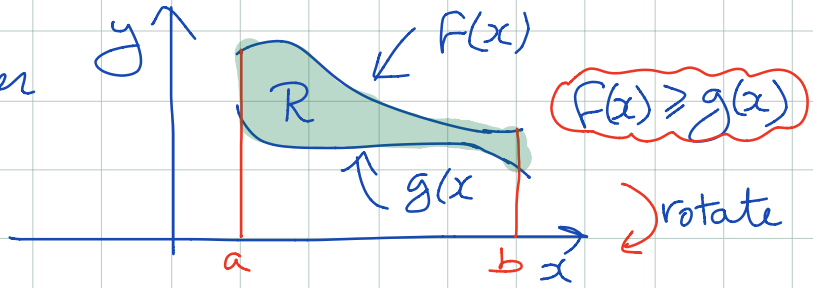


② $V = \pi \int_1^3 2x\,dx = \pi x^2 \Big|_1^3 = 9\pi - \pi = 8\pi$

~x~

Example (Region between 2 curves)

Consider the region



$F(x) \geq g(x)$

What is the volume of the region obtained by rotating R about the x-axis.

Answer: Volume = Volume generated by $f(x)$
$-$ Volume generated by $g(x)$

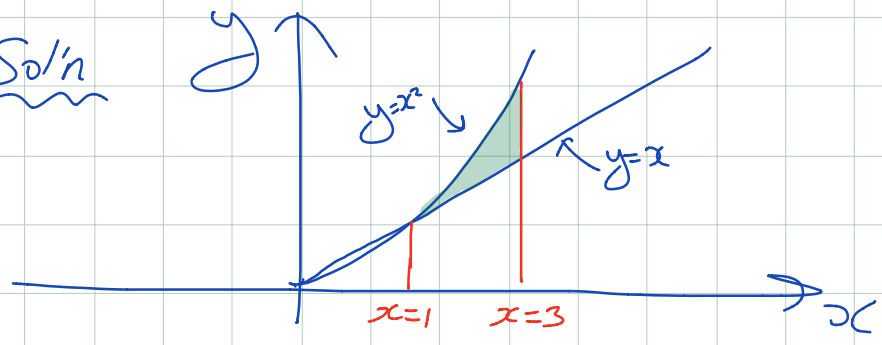
(= Outer Volume - inner Volume)

$$= \pi \int_a^b f(x)^2 \, dx - \pi \int_a^b g(x)^2 \, dx$$

so
$$V = \pi \int_a^b f(x)^2 - g(x)^2 \, dx$$

Example: Find the Volume V obtained by rotating the region between $y = x^2$ & $y = x$ about the x-axis for $1 \leq x \leq 3$

Soln



So, $V = V_{outer} - V_{inner}$

$$V = \pi \int_1^3 (x^2)^2 - x^2 \, dx$$

$$= \pi \int_1^3 x^4 - x^2 \, dx = \pi \left(\frac{x^5}{5} - \frac{x^3}{3} \right) \Big|_1^3$$

$$= \pi \left[\frac{3^5}{5} - \frac{3^3}{3} - \left(\frac{1^5}{5} - \frac{1^3}{3} \right) \right]$$

$$= \pi \left[\frac{596}{15} \right]$$

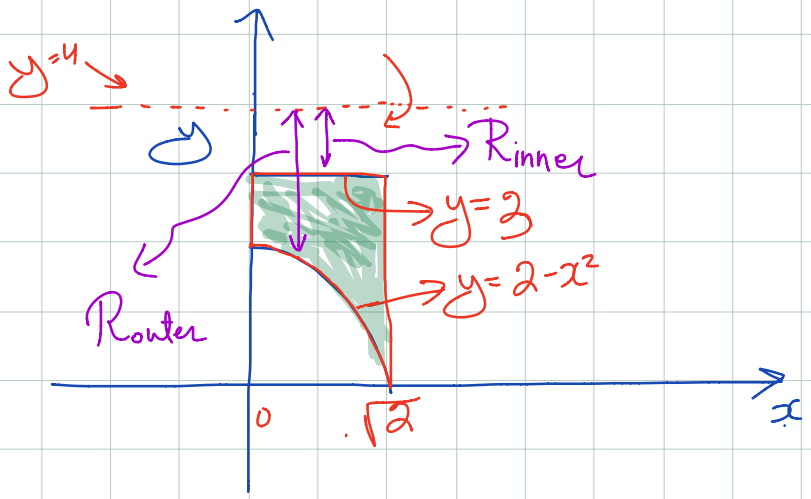
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Exercises: do the same problem with $0 \leq x \leq 3$

- Same thing, but with $f(x) = 7$, $g(x) = 3$ and $-1 \leq x \leq 2$

Example: Find the Volume obtained by rotating the region between $f(x) = 2 - x^2$ & $g(x) = 3$ about the line $x = 4$ where $0 \leq x \leq \sqrt{2}$

Soln: ① Sketch



② Volume = $V_{outer} - V_{inner}$

$$= \pi \int_0^{\sqrt{2}} R_{outer}^2 - R_{inner}^2 \, dx$$

$$= \pi \int_0^{\sqrt{2}} (4-2+x^2)^2 - (4-3)^2 \, dx$$

$$= \pi \int_0^{\sqrt{2}} 4 + x^4 + 4x^2 - 1 \, dx$$

$$= \pi \left(\frac{x^5}{5} + \frac{4}{3}x^3 + 3x \right) \Big|_0^{\sqrt{2}}$$

$$= \pi \left(\frac{4\sqrt{2}}{5} + \frac{4}{3} \cdot \frac{2\sqrt{2}}{3} + 3\sqrt{2} \right) = \text{—}$$

7.1 Integration by parts

This is the reverse product rule:

Recall that $(u(x)v(x))' = u'(x)v(x) + u(x)v'(x)$

Integrating both sides

$$\int (u(x)v(x))'\,dx = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx$$

By the FTOC:

$$u(x)v(x) = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx$$

So: $$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$$

Integration by parts

Example: Evaluate $\int x\cos x\,dx$

$$\int \underbrace{x}_{u}\ \underbrace{\cos x}_{v'}\,dx$$

Let $u(x) = x$ $\quad v'(x) = \cos(x)$

$\Rightarrow u'(x) = 1$ $\quad \Rightarrow v(x) = \sin(x)$

So $\underbrace{\int x\cos x\,dx}_{uv'} = \underbrace{x\sin x}_{uv} - \underbrace{\int \sin(x)\,dx}_{u'v}$

$= x\sin x - \cos x + C$

Exercise: Check the answer by differenting the right hand side above.

Example: Evaluate $\int xe^x\,dx$

Sol'n Same idea: let $u(x) = x$ $\quad v'(x) = e^x$

$\Rightarrow u'(x) = 1$ $\quad v(x)\ e^x$

$\Rightarrow \int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$

Tips: To integrate xe^x or $x\sin x$ or $x\cos x$ or $x^n e^x$ or $x^n \sin x$ or $x^n \cos x$

Pick $u = x$ (or x^n) and $v' = e^x$ or $\sin x$ or $\cos x$

Integration by parts more than once

Evaluate $\int x^2 \sin(x)\,dx$

Pick $u = x^2$ $\quad v' = \sin(x)$

$\Rightarrow u' = 2x$ $\quad v = \cos x$

$$\int \underbrace{x^2}_{u}\underbrace{\sin x}_{v'}\,dx = \underbrace{x^2}_{u}\underbrace{\cos x}_{v} - \int \underbrace{2x}_{u'}\underbrace{\cos x}_{v}\,dx$$

we've seen this before

have to do it by parts

$$\int x\cos x\,dx = x\sin x - \int \sin x = x\sin x - \cos x$$

So $\int x^2 \sin x\,dx = x^2\cos x - 2x\sin x + 2\cos x + C$

Example Evaluate $\int_2^7 \ln x\,dx$

Hint: use $v' = 1$

Sol'n: Let $u = \ln x$ $\quad v' = 1$

$\Rightarrow u' = 1/x$ $\quad v = x$

So $\int_2^7 \ln x\,dx = x\ln x\Big|_2^7 - \int_2^7 \frac{7}{x}\cdot x\,dx$

$= x\ln x\Big]_2^7 - x\Big]_2^7$

$= 7\ln 7 - 2\ln 2 - (7-2)$

Tip: To integrate $\int x^n \ln x\,dx$ or $\int x^n (\ln(x))^{(m)}\,dx$ (m: integer)

pick $u = \ln x$ or $\ln^m(x)$

Exercises: Evaluate
- $\int x\sin(3x)\,dx$
- $\int x^3\cos(x^2)$

Example: (Important)

Evaluate $\int e^x \sin x\,dx$

Let's try $u = \sin x$ $\quad v' = e^x$

$u' = \cos x$ & $v = e^x$

So $\int e^x \sin x\,dx = e^x \sin x - \int \cos x e^x\,dx$

by parts again!

①

$\int e^x \cos x\,dx = ?$

pick $u = \cos x \quad v' = e^x$

$u' = -\sin x \quad v = e^x$

so $\int e^x \cos x\,dx = e^x \cos x + \int e^x \sin x\,dx$

are we stuck?

②

No! Just sub ② back into ①

$$\int e^x \sin x\,dx = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx$$

So $2\int e^x \sin x\,dx = e^x \sin x - e^x \cos x$

and $\int e^x \sin x\,dx = \frac{1}{2}\left[e^x \sin x - e^x \cos x\right] + C$

Summary:

- $\int u(x)v'(x)\,dx = uv - \int u'(x)\,v(x)\,dx$
- use when you don't know how to integrate uv' but also $u'v$ is easier

Some Tips & tricks

- In general (not always) want u' to be simpler/easier than u
- Sometimes $v' = 1$ is a good choice

Example/Exercise

Show that

$$\int x^n e^x\,dx = x^n e^x - n\int x^{n-1} e^x\,dx$$